Introduction to Mathematical Quantum Theory

Solution to the Exercises

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Exercise 1

a Let $\mathcal{H}$ be an Hilbert space. Suppose $A, B \in \mathcal{B}(\mathcal{H})$ with $[A, B] = 0$ and A not invertible. Prove that AB is not invertible.

Hint: Prove first that if AB were invertible then A would have both a left and a right inverse. Then prove that those would need to be equal and conclude.

b Prove that if we do not assume A and B to commute, the result in **a** is false.

Proof. To prove **a**, suppose first that AB is invertible; this means that there is an operator C such that $ABC = \mathrm{id} = CAB$. Given that $[A, B] = 0$, we can also write $A(BC) = \mathrm{id} = (CB)A$. Now, to prove that $BC = CB$, given that A and B commute, we can write $BC = (CAB)BC = CB(ABC) = CB$. Therefore this implies that if AB is invertible then A is invertible, proving the result.

To prove **a** is enough to consider a counter example; consider A and A^* as in Exercise 3 in the Exercise Sheet of the 14.02.2014. We have that $[A, A^*] \neq 0$, both A and A^* are bounded and not invertible, but $AA^* = \mathrm{id}$, which is invertible.

□

Exercise 2

Let $\mathcal{H}$ be an Hilbert space. Let A be an unbounded linear operator on $\mathcal{H}$. Suppose there exists a closed operator C that extends the operator A. Prove that A is closable.

Proof. Recall that $G(T) := \{(\psi, T\psi) \in \mathcal{H} \times \mathcal{H} |\ \psi \in \mathcal{D}(T)\}$ is the graph of an operator T. Consider $\overline{G(A)}$; we want to prove that it corresponds to a well defined (closed) linear operator. Define the following operator:

$$\begin{aligned} \mathcal{D}(B) :&= \{\psi \in |\ \exists \varphi \in s.t.\, (\psi, \varphi) \in G(A)\} \\ B :&= C|_{\mathcal{D}(B)} . \end{aligned}$$

Given that $\mathcal{D}(A) \subseteq \mathcal{D}(B)$ is dense, we get that B is densily defined. Moreover, from the linearity of C we also get that B is linear.

From the fact that C is an extension of A we get that for any $\psi \in \mathcal{D}(A)$, $B\psi = C\psi = A\psi$, so B is an extension of A. As a consequence, $G(A) \subseteq G(B)$.

On the other hand, given that C is a closed extension of A we get that $\overline{G(A)} \subseteq \overline{G(C)} = G(C)$, so if $(\psi, \varphi) \in \overline{G(A)}$ this implies $\varphi = C\psi$. On the other hand, if $(\psi, \varphi) \in \overline{G(A)}$ then $\psi \in \mathcal{D}(B)$ and therefore $B\psi = C\psi = \varphi$ and $(\psi, \varphi) \in G(B)$. Therefore we have $\overline{G(A)} \subseteq G(B)$.

Suppose now that $(\psi, B\psi) \in G(B)$. Then given that $\psi \in \mathcal{D}(B)$ there exists an element $\varphi \in \mathcal{H}$ such that $(\psi, \varphi) \in \overline{G(A)}$; but $\overline{G(A)} \subseteq G(C)$ implies $\varphi = C\psi = B\psi$, and therefore $G(B) \subseteq \overline{G(A)}$, which together with the inclusion above shows that $G(B) = \overline{G(A)}$ and implies that A is closable.

□

Exercise 3

Let $\mathcal{H}$ be an Hilbert space. Let A be self-adjoint.

a Suppose $\lambda_0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of A. Prove that

$$\left\|(A - \lambda_0 \,\mathrm{id})^{-1}\right\| = \frac{1}{d(\lambda_0, \sigma(A))}, \tag{1}$$

where $d(x, Y) := \inf_{y \in Y} |x - y|$, with $x \in \mathbb{C}$, $Y \subseteq C$.

Hint: Think of $(A - \lambda_0 \,\mathrm{id})^{-1}$ as a function of A in the sense of the functional calculus of A.

b Let $\lambda_0 \in \mathbb{C}$ and suppose that there exists $\varepsilon > 0$ and some nonzero $\psi \in \mathcal{H}$ such that

$$\|A\psi - \lambda_0\psi\| < \varepsilon \|\psi\|. \tag{2}$$

Prove that there exists $\lambda \in \sigma(A)$ such that $|\lambda - \lambda_0| < \varepsilon$.

Proof. Recall that there exists a projection-valued measure μ^A such that

$$A = \int_{\sigma(A)} \lambda d\mu^A(\lambda),$$
$$f(A) = \int_{\sigma(A)} f(\lambda) d\mu^A(\lambda).$$

Let $\lambda_0 \in \rho(A)$; given that the spectrum of A is closed, we have $d(\lambda_0, \sigma(A)) > 0$. The function $f(\lambda) := (\lambda - \lambda_0)^{-1}$ is then continuous and bounded on $\sigma(A)$, with $\sup_{\lambda \in \sigma(A)} |f(\lambda)| = d(\lambda_0, \sigma(A))^{-1}$. Now, we know that if $g(\lambda) = \lambda - \lambda_0$, on the one hand $g(A) = A - \lambda_0 \,\mathrm{id}$ and on the other hand $g(\lambda) f(\lambda) = f(\lambda) g(\lambda) = 1$. As a consequence we get that $f(A) = (A - \lambda_0 \,\mathrm{id})^{-1}$. To get (1) then we use the functional calculus to get

$$\left\|(A - \lambda_0 \,\mathrm{id})^{-1}\right\| = \left\|\int_{\sigma(A)} f(\lambda) d\mu^A(\lambda)\right\| = \sup_{\lambda \in \sigma(A)} |f(\lambda)| = \frac{1}{d(\lambda_0, \sigma(A))}.$$

To prove **b**, assume (2); if $\lambda_0 \in \sigma(A)$, we can take $\lambda = \lambda_0$. Assume now that $\lambda_0 \in \rho(A)$. We have that

$$\left\|(A - \lambda_0 \,\mathrm{id})^{-1}\right\| \geqslant \frac{\left\|(A - \lambda_0 \,\mathrm{id})^{-1}(A - \lambda_0 \,\mathrm{id})\psi\right\|}{\|(A - \lambda_0 \,\mathrm{id})\psi\|} = \frac{\|\psi\|}{\|(A - \lambda_0 \,\mathrm{id})\psi\|} > \frac{1}{\varepsilon}.$$

Using then (1) we get

$$\frac{1}{\varepsilon} < \left\| (A - \lambda_0 \, \mathrm{id})^{-1} \right\| = \frac{1}{d\left(\lambda_0, \sigma\left(A\right)\right)},$$

which concludes the proof.

□

Exercise 4

Let $\mathcal{H} = L^2(I)$, with $I = [0, 1]$. Consider the operator A with domain $D(A) = C(I)$ and with action

$$A\psi(x) = \psi(0), \qquad \forall \psi \in D(A). \tag{3}$$

Prove that A is not closable.

Proof. Consider the graph of A given as $G(A) = \{(\psi, \psi(0)) \mid \psi \in C(I)\}$; considering $\psi = 0$, we get that $(0, 0) \in G(A)$.

Moreover, let ψ_n be a sequence of continuous functions with $\psi(I) \in [0, 1]$, $\psi(x) = 0$ for any $x \in (1/n, 1]$ and $\psi(x) = 1$ for any $x \in [0, 1/(2n))$.

Then given that $\|\psi\| \leqslant 1/n$, we get $\psi_n \to 0$ in $\mathcal{H}$ as $n \to +\infty$; on the other hand, we have that $A\psi_n(x) = 1$ for any x and for any n, so $A\psi_n \to 1$ in $\mathcal{H}$ as $n \to +\infty$. As a consequence, $(0, 1) \in \overline{G(A)}$, which implies that A is not closable.

□